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Abstract
We generalize the fractional variational problem by allowing the possibility that the lower bound in the fractional derivative does not coincide with the lower bound of the integral that is minimized. Also, for the standard case when these two bounds coincide, we derive a new form of Euler–Lagrange equations. We use approximations for fractional derivatives in the Lagrangian and obtain the Euler–Lagrange equations which approximate the initial Euler–Lagrange equations in a weak sense.

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1. Introduction

Fractional calculus with derivatives and integrals of any real or complex order has its origin in the work of Euler, and even earlier in the work of Leibnitz. Shortly after being introduced, the new theory turned out to be very attractive to many famous mathematicians and scientists (e.g. P S Laplace, B Riemann, J Liouville, N H Abel, J B J Fourier *et al*) due to the numerous possibilities for its applications. Besides mathematics, fractional derivatives and integrals appear in physics, mechanics, engineering, elasticity, dynamics, control theory, electronics, modelling, probability, finance, economics, biology, chemistry, etc. The fractional calculus is nowadays covered by several extensive reference books [15, 22, 25, 28, 33] and a large number of relevant papers.

Fractional calculus of variations unifies calculus of variations (cf classical books [13, 14, 16, 32, 35]) and fractional calculus, by inserting fractional derivatives into variational

integrals. This of course occurs naturally in many problems of physics or mechanics, in order to provide more accurate models of physical phenomena. Research within this topic goes in different directions. Jumarie [17, 18] is one of the first who has used fractional variational calculus in the analysis of fractional Brownian motion. Especially, we refer to Jumarie's paper [19] for the new approach to fractional stochastic mechanics and stochastic optimal control. We also cite Riewe [30, 31], who investigated nonconservative Lagrangian and Hamiltonian mechanics and for those cases formulated a version of the Euler–Lagrange equations. Further study of the fractional Euler–Lagrange equations can be found in the work of Agrawal [1–3], who considered different types of variational problems, involving Riemann–Liouville, Caputo and Riesz fractional derivatives, respectively. He derived corresponding Euler–Lagrange equations and discussed possibilities for prescribing boundary conditions in each case. The work of mentioned authors influenced many recent papers. For instance, Baleanu [7, 8] applied the fractional Euler–Lagrange equations to examine fractional Lagrangian and Hamiltonian systems linear in velocities. Other applications of fractional variational principles are presented in [4–6, 9, 10, 12, 23, 24, 26, 27, 29]. We also cite here the work of Frederico and Torres [11], who introduced a new concept of fractional conserved quantities on the basis of a variational principle, and proved a version of the fractional Nöther theorem.

There are several aims of this paper. First, we discuss the Euler–Lagrange equations of [1] and [2] and show that the transversality condition proposed in [2] should be used with care, since it may lead to erroneous conclusions. Second, we consider a fractional variational problem, defined by a functional whose lower bound does not coincide with the lower bound in the left Riemann–Liouville fractional derivative that appears in the Lagrangian. This leads to the natural generalization of the fractional variational problems considered so far. Third, in section 4 we approximate fractional derivative in the Lagrangian $L\left(t, u(t), {}_aD_t^\alpha u\right)$ with a finite number of terms containing derivatives of integer order, which reduces the variational problem to the one depending only on the classical derivatives of the function u. For that purpose we consider approximations in a weak sense, using analytic functions as a test function space, and show that a sequence of approximated Euler–Lagrange equations converges to the fractional Euler–Lagrange equation.

2. Notation

Let $u \in L^1([a, b])$ and $0 \leqslant \alpha, \beta < 1$. Then the left Riemann–Liouville fractional integral of order α, ${}_aI_t^\alpha u$ is defined as

$$
{}_aI_t^\alpha u = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \theta)^{\alpha-1} u(\theta) \, d\theta, \qquad t \in [a, b].
$$

The right Riemann–Liouville fractional integral of order β, ${}_tI_b^\beta u$ is defined as

$$
{}_tI_b^\beta u = \frac{1}{\Gamma(\beta)} \int_t^b (\theta - t)^{\beta-1} u(\theta) \, d\theta, \qquad t \in [a, b].
$$

If u is an absolutely continuous function in $[a, b]$, i.e. $u \in AC([a, b])$, and $0 \leqslant \alpha < 1$, then the left Riemann–Liouville fractional derivative of order α, ${}_aD_t^\alpha u$ is given by

$$
{}_aD_t^\alpha u = \frac{d}{dt} {}_aI_t^{1-\alpha} u = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_a^t (t - \theta)^{-\alpha} u(\theta) \, d\theta, \qquad t \in [a, b],
$$

and for $0 \leqslant \beta < 1$, the right Riemann–Liouville fractional derivative of order β, ${}_tD_b^\beta u$ is given by

$$
{}_tD_b^\beta u = \left(-\frac{d}{dt}\right) {}_tI_b^{1-\beta} u = \frac{1}{\Gamma(1 - \beta)} \left(-\frac{d}{dt}\right) \int_t^b (\theta - t)^{-\beta} u(\theta) \, d\theta, \qquad t \in [a, b].
$$

We have that $_aD_t^\alpha {}_aI_t^\alpha = I$, where I is the identity map. The formula for fractional integration by parts reads (see [33, p 46])

$$\int_a^b f(t) {}_aD_t^\alpha g \, \mathrm{d}t = \int_a^b g(t) {}_tD_b^\alpha f \, \mathrm{d}t, \qquad f, g \in AC([a, b]). \tag{1}$$

In the distributional setting, the Riemann–Liouville fractional derivatives can be defined via convolutions in the space of tempered distributions supported by $[0, +\infty)$. Let $(f_\alpha)_{\alpha \in \mathbb{R}} \in \mathcal{S}'_+ = \{f \in \mathcal{S}'(\mathbb{R}) : \operatorname{supp} f \subset [0, +\infty)\}$ be a family of distributions defined as

$$f_\alpha(t) = \begin{cases} H(t) \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0 \\ f_{\alpha+N}^{(N)}(t), & N \in \mathbb{N} : N + \alpha > 0 \wedge N + \alpha - 1 < 0, \end{cases}$$

where H is the Heaviside function. The operator $f_\alpha *$ is the Riemann–Liouville operator of differentiation, resp. integration of order α for $\alpha < 0$, resp. $\alpha > 0$. In this setting $_aI_t^\alpha$ and $_aD_t^\alpha$ are inverses in both directions, due to the group property in $\mathcal{S}'$, that is $f_\alpha * f_\beta = f_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{R}$.

Beside the Riemann–Liouville approach there exist several other possibilities for introducing derivatives of fractional order. We will make use of the Caputo fractional derivatives: if $0 \leqslant \alpha, \beta < 1$ and $u \in AC([a, b])$, then the left Caputo fractional derivative of order α, $_a^cD_t^\alpha u$ is defined as

$$_a^cD_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\theta)^{-\alpha} \dot{u}(\theta) \, \mathrm{d}\theta, \qquad t \in [a, b],$$

where $(\dot{\cdot})$ denotes the total derivative $\frac{\mathrm{d}}{\mathrm{d}t}$, and the right Caputo fractional derivative of order β, $_t^cD_b^\beta u$ is defined as

$$_t^cD_b^\beta u = -\frac{1}{\Gamma(1-\beta)} \int_t^b (\theta-t)^{-\beta} \dot{u}(\theta) \, \mathrm{d}\theta, \qquad t \in [a, b].$$

The Riemann–Liouville and Caputo fractional derivatives are related by the following formula:

$$_aD_t^\alpha u = {}_a^cD_t^\alpha u + \frac{1}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^\alpha}, \qquad t \in [a, b], \tag{2}$$

and similarly

$$_tD_b^\beta u = {}_t^cD_b^\beta u + \frac{1}{\Gamma(1-\beta)} \frac{u(b)}{(b-t)^\alpha}, \qquad t \in [a, b]. \tag{3}$$

If $u(a) = 0$ (resp. $u(b) = 0$) then the left (resp. right) Riemann–Liouville and Caputo fractional derivatives coincide. Also, for $u(a) = 0$ the left Riemann–Liouville fractional derivative commutes with the first derivative with respect to t, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} {}_aD_t^\alpha u = {}_aD_t^\alpha \frac{\mathrm{d}}{\mathrm{d}t} u$ (and the same holds for the right Riemann–Liouville fractional derivative if $u(b) = 0$).

In this paper, we will consider the Lagrangian L as a function of t, u and $_aD_t^\alpha u$, i.e. $L = L(t, u(t), {}_aD_t^\alpha u)$. The partial derivatives of L will be denoted by $\frac{\partial L}{\partial t}$, $\frac{\partial L}{\partial u}$ and $\frac{\partial L}{\partial {}_aD_t^\alpha u}$ or by $\partial_1 L$, $\partial_2 L$ and $\partial_3 L$, respectively. The first (or Lagrangian) variation will be denoted by δ, as usual.

3. Euler–Lagrange equations

Let (A, B) be a subinterval of (a, b). Consider a functional

$$\mathcal{L}[u] = \int_A^B L(t, u(t), {}_aD_t^\alpha u) \, \mathrm{d}t, \qquad 0 \leqslant \alpha < 1, \tag{4}$$

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where u is an absolutely continuous function in $[a, b]$ and L is a function in $(a, b) \times \mathbb{R} \times \mathbb{R}$ such that

$$\left. \begin{array}{c} L \in \mathcal{C}^1((a, b) \times \mathbb{R} \times \mathbb{R}) \\ \text{and} \\ t \mapsto \partial_2 L\big(t, u(t), {}_aD_t^\alpha u\big) \text{ is integrable in } (a, b) \text{ and} \\ t \mapsto \partial_3 L\big(t, u(t), {}_aD_t^\alpha u\big) \in AC[a, b], \text{ for every } u \in AC([a, b]) \end{array} \right\}. \qquad (5)$$

A fractional variational problem consists of finding extremal values (minima or maxima) of the functional (4) among all admissible functions. The function L is called the Lagrangian. Note that in (4) the constants a and A are assumed to be different and in general $a \leqslant A$. Their physical meaning is also different. While the interval (A, B) defines the Hamilton action, the value a defines memory of the system. In the special case, that was treated previously (cf [1, 2, 7, 8, 11]), it was assumed that $a = A$.

As mentioned above, $\mathcal{L}$ is to be minimized (or maximized) over the set of admissible functions. Hence, we have to specify where we look for a minimum of (4): the admissible set will consist of all absolutely continuous functions u in $[a, b]$, which pass through a fixed point at a, i.e. $u(a) = a_0$, for a fixed $a_0 \in \mathbb{R}$.

Remark 3.1.

  (i) We consider a fractional variational problem which involves only left Riemann–Liouville fractional derivatives. The problem can be easily generalized for Lagrangians which will depend also on the right Riemann–Liouville fractional derivatives.
 (ii) We assume that $0 \leqslant \alpha < 1$. Our assumption can be extended to the case $\alpha \geqslant 1$ without difficulties.
(iii) Traditionally, the minimizers of a variational problem are sought. Analogously, one can consider the problem of finding the maximal values of a variational problem.

In this section, we discuss the results on the Euler–Lagrange equations of the fractional variational problem (4).

The case $A = a$ was treated in [1] by Agrawal. It was proved there that if one wants to minimize (4) among all functions u which have continuous left αth Riemann–Liouville fractional derivative and which satisfy the Dirichlet boundary conditions $u(a) = a_0$ and $u(b) = b_0$, for some real constant values a_0 and b_0, then a minimizer should be sought among all solutions of the Euler–Lagrange equation

$$\frac{\partial L}{\partial u} + {}_tD_b^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) = 0. \qquad (6)$$

This result was modified in [2], where again $a = A$ was used, and the boundary condition was specified at $t = a$ only, which allowed the natural boundary conditions to be developed. The corresponding Euler–Lagrange equation was obtained as

$$\frac{\partial L}{\partial u} + {}_t^cD_b^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) = 0, \qquad (7)$$

with the transversality (natural) condition

$$\frac{\partial L}{\partial_a D_t^\alpha u} {}_aI_t^{1-\alpha} \delta u = 0 \qquad \text{at } t = b. \qquad (8)$$

It is clear that (7) and (8) imply (6). But the converse does not hold in general. This depends on assumptions on L and the set of admissible functions. Our first example which is to follow will show that assuming (5), condition (6) does not imply (7) and (8).

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Example 3.2. Consider a fractional variational problem of the form

$$\mathcal{L}[u] = \int_0^1 L\big(t, u(t), {}_0D_t^\alpha u\big)\, \mathrm{d}t \rightarrow \min \tag{9}$$

with $u \in AC([0, 1])$ and L to be specified.

(i) Let the Lagrangian L satisfies (5) and has the form

$$L\big(t, u, {}_aD_t^\alpha u\big) = F(t, u) + f(t){}_aD_t^\alpha u.$$

Then $\partial L/\partial_a D_t^\alpha u = f(t), t \in (0, 1)$, and we have (see [2, (13)]) that

$$0 = \int_0^1 \left(\frac{\partial F}{\partial u} + {}_t^c D_1^\alpha f \right) \delta u\, \mathrm{d}t + f(t){}_0D_t^{\alpha-1}\delta u \bigg|_0^1$$

$$= \int_0^1 \left(\frac{\partial F}{\partial u} + \frac{1}{\Gamma(1-\alpha)} \int_t^1 \frac{f'(\theta)}{(\theta - t)^\alpha}\, \mathrm{d}\theta + \frac{f(1)}{\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha} \right) \delta u\, \mathrm{d}t.$$

Now, take for example $F = \frac{u^2}{2\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha}$ and $f(t) = -1, t \in (0, 1)$. Then the Euler–Lagrange equation (6) gives

$$\frac{u}{\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha},$$

thus $u \equiv 1$ in $[0, 1]$. Hence, if one wants to formulate (9) so that (6) holds if and only if (7) and (8) hold, then some additional assumptions on F have to be supposed. For instance, the condition $f(1) = 0$ provides the desired equivalence of (6) with (7, 8).

(ii) The Lagrangian

$$L\big(t, u, {}_aD_t^\alpha u\big) = \big({}_aD_t^\alpha u - u\big)^2$$

was investigated in detail in [2]. Solutions of the corresponding fractional variational problem (9) can be found directly. It is clear that the functional $\mathcal{L}$ achieves its minimum (which is zero) when ${}_aD_t^\alpha u - u = 0$. Hence, the problem reduces to solving the equation ${}_aD_t^\alpha u = u$. It was shown in [22, p 222] that this equation has no solution which is bounded at 0. In other words, one cannot solve the fractional variational problem with the Lagrangian given above among the functions with the prescribed, finite boundary condition at 0. If instead of the left Riemann–Liouville fractional derivative one considers the Lagrangian L as a function of the left Caputo fractional derivative, then the equation ${}_a^c D_t^\alpha u = u$ has a solution bounded at zero, which is also a solution of the corresponding fractional variational problem (9). In a recent paper [3], the author tried to overcome the problem of non-solvability of the equation ${}_aD_t^\alpha u = u$ by using the symmetrized Caputo fractional derivative, called the Riesz Caputo fractional derivative and defined as ${}_a^{rc}D_t^\alpha u := \frac{1}{2}\big({}_a^c D_t^\alpha u - {}_t^c D_b^\alpha u\big)$. Similar kind of fractional derivatives were used earlier in [23, 24].

These examples suggest that the Euler–Lagrange equation for the fractional variational problem (4) with $A = a$ and with the boundary condition specified at $t = a$, should be reformulated as follows:

$$\frac{\partial L}{\partial u} + {}_t^c D_b^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) + \frac{\partial L}{\partial_a D_t^\alpha u} \bigg|_{t=b} \frac{1}{\Gamma(1-\alpha)} \frac{1}{(b-t)^\alpha} = 0, \tag{10}$$

instead of (7) and (8).

We present now the Euler–Lagrange equation for (4). We state this as

Theorem 3.3. *Let $u^* \in AC([a, b])$ be an extremal of the functional $\mathcal{L}$ in (4), whose Lagrangian L satisfies (5). Then u^* satisfies the following Euler–Lagrange equations*

$$\frac{\partial L}{\partial u} + {}_t^c D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) + \frac{\partial L}{\partial_a D_t^\alpha u} \Big|_{t=B} \frac{1}{\Gamma(1-\alpha)} \frac{1}{(B-t)^\alpha} = 0, \qquad t \in (A, B) \tag{11}$$

$$_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) - {}_t D_A^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) = 0, \qquad t \in (a, A). \tag{12}$$

Proof. It is known that a necessary condition for a solution u^* of a fractional variational problem $\mathcal{L}[u]$ defined by (4) is that the first variation of $\mathcal{L}[u]$ is zero at the solution u^*, i.e.

$$\begin{aligned}
0 &= \delta \mathcal{L}[u] \\
&= \int_A^B \delta L\big(t, u(t), {}_a D_t^\alpha u\big) \, dt \\
&= \int_A^B \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \big[L\big(t, u(t) + \varepsilon \delta u(t), {}_a D_t^\alpha (u + \varepsilon \delta u)\big) - L\big(t, u(t), {}_a D_t^\alpha u\big) \big] \, dt \\
&= \int_A^B \left[\frac{\partial L}{\partial u} \delta u(t) + \frac{\partial L}{\partial_a D_t^\alpha u} {}_a D_t^\alpha \delta u(t) \right] dt,
\end{aligned} \tag{13}$$

where δu is the Lagrangian variation of u, i.e. $\delta u(a) = 0$. Integration by parts formula (1) gives that

$$\int_a^B \frac{\partial L}{\partial_a D_t^\alpha u} {}_a D_t^\alpha \delta u(t) \, dt = \int_a^B \delta u(t) {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) dt.$$

Thus we obtain

$$\begin{aligned}
\int_a^B \frac{\partial L}{\partial_a D_t^\alpha u} {}_a D_t^\alpha \delta u(t) \, dt &= \int_A^B \frac{\partial L}{\partial_a D_t^\alpha u} {}_a D_t^\alpha \delta u(t) \, dt + \int_a^A \frac{\partial L}{\partial_a D_t^\alpha u} {}_a D_t^\alpha \delta u(t) \, dt \\
&= \int_A^B \delta u(t) {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) dt + \int_a^A \delta u(t) {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) dt.
\end{aligned}$$

From the last equality we conclude that

$$\begin{aligned}
\int_A^B \frac{\partial L}{\partial_a D_t^\alpha u} {}_a D_t^\alpha \delta u(t) \, dt &= \int_A^B \delta u(t) {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) dt + \int_a^A \delta u(t) {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) dt \\
&\quad - \int_a^A \frac{\partial L}{\partial_a D_t^\alpha u} {}_a D_t^\alpha \delta u(t) \, dt \\
&= \int_A^B \delta u(t) {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) dt \\
&\quad + \int_a^A \left[{}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) - {}_t D_A^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) \right] \delta u(t) \, dt.
\end{aligned}$$

If we insert this into (13) we obtain

$$0 = \int_A^B \left[\frac{\partial L}{\partial u} + {}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) \right] \delta u(t) \, dt + \int_a^A \left[{}_t D_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) - {}_t D_A^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) \right] \delta u(t) \, dt.$$

Therefore,

$$\frac{\partial L}{\partial u} + {}_t D_B^\alpha \frac{\partial L}{\partial_a D_t^\alpha u} = 0, \qquad t \in (A, B)$$

6

and

$$
{}_tD_B^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) - {}_tD_A^\alpha \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right), \qquad t \in (a, A).
$$

The claim now follows if we replace the right Riemann–Liouville by the right Caputo fractional derivative according to (3) in the first equation. □

Remark 3.4. It is interesting to compare the Euler–Lagrange equations (10) and (11), (12) in the case $B = b$, when $A > a$ and $A \to a$. Thus, if we let $A \to a$ in (11), (12) we obtain the Euler–Lagrange equation (10) plus an additional condition

$$
{}_AI_B^{1-\alpha} \left(\frac{\partial L}{\partial_a D_t^\alpha u} \right) \equiv \text{const.}
$$

Indeed, since

$$
0 = {}_tD_B^\alpha(\partial_3 L) - {}_tD_A^\alpha(\partial_3 L) = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\Gamma(1-\alpha)} \int_A^B \frac{\partial_3 L(\theta, u(\theta), {}_aD_\theta^\alpha u)}{(\theta - t)^\alpha} \, \mathrm{d}\theta, \qquad t \in (a, A),
$$

we obtain that ${}_AI_B^{1-\alpha}(\partial_3 L) \equiv \text{const.}$

Remark 3.5. Comparing fractional calculus used in [19–21] and here, one can note the difference in approaches: in his work Jumarie used a modified Riemann–Liouville fractional derivative that has different properties (see, e.g. [20, 21]).

Remark 3.6. The problem of formulating necessary and sufficient conditions which guarantee that given fractional order differential equation is derivable from a variational principle is still open (cf [34] for inverse problems with integer order derivatives). However, it is shown in [9, 10] that a necessary condition for constructing a fractional Lagrangian from the given Euler–Lagrange equation is that the Euler–Lagrange equation involves both left and right Riemann–Liouville fractional derivatives. If only one of them appears then such an equation cannot be the Euler–Lagrange equation for some Lagrangian. For example, according to [9, 10] it is not possible to construct a fractional Lagrangian for a linear oscillator with fractional derivative $u'' + u + {}_aD_t^\alpha u = 0$, and therefore the same holds for a nonlinear oscillator of the type $u'' + f(u) + {}_aD_t^\alpha u = 0$.

4. Approximation of Euler–Lagrange equations

In this section, we use the approximation of the Riemann–Liouville fractional derivative by the finite sum where derivatives of integer order appear, and in this way we analyze a fractional variational problem involving only classical derivatives. Then we examine relation between the Euler–Lagrange equations obtained in the process of approximations and the fractional Euler–Lagrange equations derived in the previous section.

We will assume in the following that $L \in \mathcal{C}^N([a, b] \times \mathbb{R} \times \mathbb{R})$, at least.

Let (c, d), $-\infty < c < d < +\infty$, be an open interval in $\mathbb{R}$ which contains $[a, b]$, such that for each $t \in [a, b]$ the closed ball $L(t, b - a)$, with centre at t and radius $b - a$, lies in (c, d).

For any real analytic function f in (c, d) we can write the following expansion formula:

$$
{}_aD_t^\alpha f = \sum_{i=0}^\infty \binom{\alpha}{i} \frac{(t - a)^{i-\alpha}}{\Gamma(i + 1 - \alpha)} f^{(i)}(t), \qquad t \in L(t, b - a) \subset (c, d), \quad (14)
$$

where $\binom{\alpha}{i} = \frac{(-1)^{i-1}\alpha\Gamma(i-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)}$ (cf [33, (15.4) and (1.48)]). Actually, condition $L(t, b-a) \subset (c, d)$ is not formulated in the literature; it comes from the Taylor expansion of $f(t - \tau)$ at t, for $\tau \in (a, t)$ and $t \in (a, b)$.

Consider again the fractional variational problem (4). Assume that we are looking for a minimizer $u \in \mathcal{C}^{2N}([a,b])$, for some $N \in \mathbb{N}$. We replace in the Lagrangian the left Riemann–Liouville fractional derivative $_a D_t^\alpha u$ by the finite sum of integer-valued derivatives as in (14):

$$\int_A^B L\left(t, u(t), \sum_{i=0}^N \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}(t)\right) dt$$

$$= \int_A^B \bar{L}(t, u(t), u^{(1)}(t), u^{(2)}(t), \ldots, u^{(N)}(t)) dt. \tag{15}$$

Now the Lagrangian $\bar{L}$ depends on t, u and all (classical) derivatives of u up to order N. Moreover, $\partial_3 \bar{L}, \ldots, \partial_{N+2} \bar{L} \in \mathcal{C}^{N-1}([a,b] \times \mathbb{R} \times \mathbb{R})$, since $\partial_i \bar{L} = \partial_3 L \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}, i = 3, \ldots, N+2$.

The Euler–Lagrange equation for (15) has the following form:

$$\sum_{i=0}^N \left(-\frac{d}{dt}\right)^i \frac{\partial \bar{L}}{\partial u^{(i)}} = 0.$$

This is equivalent to

$$\frac{\partial L}{\partial u} + \sum_{i=0}^N \left(-\frac{d}{dt}\right)^i \left(\partial_3 L \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) = 0. \tag{16}$$

Remark 4.1. The Euler–Lagrange equation (16) provides a necessary condition when one solves the variational problem (15) in the class $\mathcal{C}^{2N}([a,b])$, with the prescribed boundary condition at A and B, i.e. $u(A) = A_0$ and $u(B) = B_0$, A_0, B_0 are fixed real numbers.

The question arises how (16) is related to (11), (12). More precisely, we want to show that (16) converges to (11), (12), as $N \to \infty$, in a weak sense.

We will simplify the proof by considering the case $A = a$ and $B = b$. This choice reduce (11), (12) to (10).

First we prove a result which provides an expression for the right Riemann–Liouville fractional derivative in terms of the lower bound a, which figures in the left Riemann–Liouville fractional derivative. Such an equality holds in a weak sense, if for a test function space we use the space of real analytic functions as follows.

Let $\mathcal{A}((c,d))$ be the space of real analytic functions in (c,d) with the family of seminorms

$$p_{[m,n]}(\varphi) := \sup_{t \in [m,n]} |\varphi(t)|, \qquad \varphi \in \mathcal{A}((c,d)),$$

where $[m,n]$ are subintervals of (c,d). Every function $f \in \mathcal{C}([a,b])$, which we extend to be zero in $(c,d)\setminus[a,b]$, defines an element of the dual $\mathcal{A}'((c,d))$ via

$$\varphi \mapsto \langle f, \varphi \rangle = \int_a^b f(t)\varphi(t) dt, \qquad \varphi \in \mathcal{A}((c,d)).$$

As usual, we say that f and g from $\mathcal{A}'((c,d))$ are equal in the weak sense if for every $\varphi \in \mathcal{A}((c,d))$,

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle.$$

In the proposition and theorem which are to follow, we will assume (as in (14)) that $L(t, b-a) \subset (c,d)$, for all $t \in [a,b]$.

Proposition 4.2. *Let* $F \in \mathcal{C}^\infty([a,b])$, *such that* $F^{(i)}(b) = 0$, *for all* $i \in \mathbb{N}_0$, *and* $F \equiv 0$ *in* $(c,d)\setminus[a,b]$. *Let* $_t D_b^\alpha F$ *be extended by zero in* $(c,d)\setminus[a,b]$. *Then:*

(i) *For every $i \in \mathbb{N}$, the $(i-1)$th derivative of $t \mapsto F(t)(t-a)^{i-\alpha}$ is continuous at $t = a$ and $t = b$ and the ith derivative of this function, $i \in \mathbb{N}_0$, is integrable in (c, d) and supported by $[a, b]$.*

(ii) *The partial sums S_N, $N \in \mathbb{N}_0$, where*

$$
t \mapsto S_N(t) := \begin{cases} \sum_{i=0}^{N} \left(-\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F \cdot \binom{\alpha}{i} \dfrac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right), & t \in [a, b], \\ 0, & t \in (c, d) \setminus [a, b], \end{cases}
$$

are integrable functions in (c, d) and supported by $[a, b]$;

(iii)

$$
{}_t D_b^\alpha F = \sum_{i=0}^{\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) \tag{17}
$$

in the weak sense.

Proof. One can simply prove the assertions (i) and (ii) concerning the mappings $t \mapsto F(t)(t-a)^{i-\alpha}$ and $t \mapsto S_N(t), t \in [a, b]$.

So, let us prove the main assertion (iii). We have to show that

$$
\langle {}_t D_b^\alpha F, \varphi \rangle = \left\langle \sum_{i=0}^{\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right), \varphi \right\rangle, \qquad \forall \varphi \in \mathcal{A}((c, d)).
$$

Since ${}_t D_b^\alpha F$ is continuous in $[a, b]$, it follows:

$$
\begin{aligned}
\langle {}_t D_b^\alpha F, \varphi \rangle &= \int_a^b {}_t D_b^\alpha F \varphi(t) \, \mathrm{d}t \\
&= \int_a^b F(t) {}_a D_t^\alpha \varphi \, \mathrm{d}t,
\end{aligned}
$$

where we have used fractional integration by parts (1). Now, by (14), (i) and (ii), we continue

$$
\begin{aligned}
\langle {}_t D_b^\alpha F, \varphi \rangle &= \int_a^b F(t) \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \varphi^{(i)}(t) \, \mathrm{d}t \\
&= \lim_{N \to \infty} \left\langle F(t), \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \varphi^{(i)}(t) \right\rangle \\
&= \lim_{N \to \infty} \int_a^b \sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F(t) \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) \varphi(t) \, \mathrm{d}t.
\end{aligned}
$$

This implies that

$$
\lim_{N \to \infty} \sum_{i=0}^{N} \int_a^b \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F(t) \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right)
$$

exists in $\mathcal{A}'((c, d))$ and

$$
\begin{aligned}
\lim_{N \to \infty} \int_a^b \sum_{i=0}^{N} &\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F(t) \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) \varphi(t) \, \mathrm{d}t \\
&= \left\langle \sum_{i=0}^{\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(F \cdot \binom{\alpha}{i} \frac{(\cdot - a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right), \varphi \right\rangle.
\end{aligned}
$$

This proves (17). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will show in the theorem which is to follow, that the Euler–Lagrange equation (16) converges to (10), as $N \to +\infty$, in the weak sense. To shorten the notation, we introduce P_N and P for the Euler–Lagrange equations in (16) and (10), respectively.

We will use the following assumptions:

(a) Let $u \in \mathcal{C}^{\infty}([a, b])$ such that $u(a) = a_0, u(b) = b_0$, for fixed $a_0, b_0 \in \mathbb{R}$, and L_3 (where L_3 stands for $\partial_3 L$) be a function in $[a, b]$ defined by $t \mapsto L_3(t) = L_3(t, u(t), {}_aD_t^\alpha u), t \in [a, b]$. Let $L_3^{(i)}(b, b_0, p) = 0$, for all $i \in \mathbb{N}$, meaning that for $(t, s, p) \mapsto L_3(t, s, p), t \in [a, b], s, p \in \mathbb{R}$, the following holds:

   (i) $\dfrac{\partial^i L_3}{\partial t^i}(b, b_0, p) = 0, \qquad \forall p \in \mathbb{R};$

   (ii) $\dfrac{\partial^i L_3}{\partial s^i}(b, b_0, p) = 0, \qquad \forall p \in \mathbb{R};$

   (iii) $\dfrac{\partial^i L_3}{\partial p^i}(b, b_0, p) = 0, \qquad \forall p \in \mathbb{R}.$

(b) Let $u \in \mathcal{C}^{\infty}([a, b])$ such that $u^{(i)}(b) = 0$, for all $i \in \mathbb{N}_0$, and $u(a) = a_0$, for fixed $a_0 \in \mathbb{R}$. Let $L_3^{(i)}(b) = L_3^{(i)}(b, 0, {}_aD_b^\alpha u) = 0$, for all $i \in \mathbb{N}$ and for every fixed u, meaning that for $(t, s, p) \mapsto L_3(t, s, p), t \in [a, b], s, p \in \mathbb{R}$, the following holds:

   (i) $\dfrac{\partial^i L_3}{\partial t^i}(b, 0, p) = 0, \qquad \forall p \in \mathbb{R};$

   (ii) $\dfrac{\partial^i L_3}{\partial p^i}(b, 0, p) = 0, \qquad \forall p \in \mathbb{R}.$

Now we will consider the fractional variational problem (4) in the case (a) and in the case (b).

Theorem 4.3. *Let $\mathcal{L}[u]$ be a fractional variational problem (4) which is being solved in the case (a) or (b). Denote by P the fractional Euler–Lagrange equations (11), and by P_N the Euler–Lagrange equations (16), which correspond to the variational problem (15), in which the left Riemann–Liouville fractional derivative is approximated according to (14) by the finite sum. Then in both cases (a) and (b)*

$$P_N \to P \text{ in the weak sense, as } N \to 0.$$

Proof. The proof of the theorem is based on proposition 4.2. By assumptions (a) and (b) and the extensions of partial derivatives of L to be zero in $(c, d) \backslash [a, b]$ we can apply Proposition 4.2 with $F(t) = \partial_3 L(t, u(t), {}_aD_t^\alpha u), t \in [a, b]$ ($F \equiv 0$ in $(c, d) \backslash [a, b]$). For any $\varphi \in \mathcal{A}((c, d))$ the following holds:

$$\lim_{N \to +\infty} \left\langle \frac{\partial L}{\partial u}(t, u(t), {}_aD_t^\alpha u) + \sum_{i=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \left(\partial_3 L(t, u(t), {}_aD_t^\alpha u) \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right), \varphi(t) \right\rangle$$

$$= \left\langle \frac{\partial L}{\partial u}, \varphi \right\rangle + \lim_{N \to +\infty} \left\langle \partial_3 L, \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(\cdot - a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \varphi^{(i)} \right\rangle$$

$$= \left\langle \frac{\partial L}{\partial u}, \varphi \right\rangle + \left\langle \partial_3 L, {}_aD_t^\alpha \varphi \right\rangle$$

$$= \left\langle \frac{\partial L}{\partial u} + {}_tD_b^\alpha \frac{\partial L}{\partial {}_aD_t^\alpha u}, \varphi \right\rangle.$$

The claim now follows from (2). □

5. Concluding remarks

Euler–Lagrange equations have been studied for a general fractional variational problem in which the lower bound in the variational integral differs from the lower bound in the left Riemann–Liouville fractional derivative which appears in the Lagrangian. Thus we allow for the possibility that the beginning of the memory of the system ($t = a$) does not coincide with the lower bound ($t = A$) in the Hamiltonian's action integral. Also, the previous results related to fractional Euler–Lagrange equations have been corrected and improved.

An approximation of fractional derivatives in the Lagrangian has been suggested, resulting in a derivation of approximate Euler–Lagrange equations. Since the Leibnitz formula does not hold for $_aD_t^\alpha(f \cdot g)$, the passage from the approximated to fractional Euler–Lagrange equations is done by the use of weak limits over a specified test function space. In this way right and left Riemann–Liouville fractional derivatives are related to each other in a weak sense.

The further research will continue towards fractional variational symmetries and Nöther's theorem. In this context theorem 4.3 has an important role, since in a similar manner will be approximated the corresponding infinitesimal criterion as well as Nöther's theorem, which leads to a further analysis of variational symmetries through fractional calculus.

References

[1] Agrawal O P 2002 Formulation of Euler-Lagrange equations for fractional variational problems *J. Math. Anal. Appl.* **272** 368–79
[2] Agrawal O P 2006 Fractional variational calculus and the transversality conditions *J. Phys. A: Math. Gen.* **39** 10375–84
[3] Agrawal O P 2007 Fractional variational calculus in terms of Riesz fractional derivatives *J. Phys. A: Math. Theor.* **40** 6287–303
[4] Atanacković T M, Oparnica Lj and Pilipović S 2007 On a nonlinear distributed order fractional differential equation *J. Math. Anal. Appl.* **328** 590–608
[5] Atanacković T M and Stanković B 2007 On a class of differential equations with left and right fractional derivatives *Z. Angew. Math. Mech.* **87** 537–46
[6] Baleanu D 2006 Fractional Hamiltonian analysis of irregular systems *Signal Process.* **86** 2632–6
[7] Baleanu D and Avkar T 2004 Lagrangians with linear velocities within Riemann-liouville fractional derivatives *Il Nuovo Cimento* B **119** 73–9
[8] Baleanu D and Muslih S I 2005 Hamiltonian formulation of systems with linear velocities within Riemann-liouville fractional derivatives *J. Math. Anal. Appl.* **304** 599–606
[9] Dreisigmeyer D W and Young P M 2003 Extending Bauer's corollary to fractional derivatives *J. Phys. A: Math. Gen.* **37** 117–21
[10] Dreisigmeyer D W and Young P M 2003 Nonconservative Lagrangian mechanics: a generalized function approach *J. Phys. A: Math. Gen.* **36** 8297–310
[11] Frederico G S F and Torres D F M 2007 A formulation of Nöther's theorem for fractional problems of the calculus of variations *J. Math. Anal. Appl.* **334** 834–46
[12] Gaies A and El-Akrmi A 2004 Fractional variational principle in macroscopic picture *Phys. Scr.* **70** 7–10
[13] Gelfand I M and Fomin S V 2000 *Calculus of Variations* (New York: Dover)
[14] Giaquinta M and Hildebrandt S 1996 *Calculus of Variations I* (Berlin: Springer)
[15] Gorenflo R and Mainardi F 1997 Fractional calculus: integral and differential equations of fractional order *Fractals and Fractional Calculus in Continuum Mechanics* ed F Mainardi and A Carpinteri (CISM Courses and Lectures vol 378) (New York: Springer) pp 223–76
[16] Jost J and Li-Jost X 1998 *Calculus of Variations* (Cambridge: Cambridge University Press)
[17] Jumarie G 1993 Stochastic differential equations with fractional Brownian motion input *Int. J. Syst. Sci.* 1113–32
[18] Jumarie G 1995 A practical variational approach to stochastic optimal control via state moment equations *J. Franklin Inst.: Eng. Appl. Math.* 761–72
[19] Jumarie G 2006 Fractionalization of the complex-valued Brownian motion of order *n* using Riemann–Liouville derivative. Applications to mathematical finance and stochastic mechanics *Chaos Solitons Fractals* 1285–305

[20] Jumarie G 2006 Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results *Comput. Math. Appl.* 1367–76

[21] Jumarie G 2007 Lagrangian mechanics of fractional order, Hamilton–Jacobi fractional pde and Taylor's series of nondifferential functions *Chaos Solitons Fractals* 969–87

[22] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* (Amsterdam: Elsevier)

[23] Klimek M 2001 Fractional sequential mechanics—model with symmetric fractional derivative *Czech. J. Phys.* **51** 1348–54

[24] Lazopoulos K A 2006 Non-local continuum mechanics and fractional calculus *Mech. Res. Commun.* **33** 753–7

[25] Miller K S and Ross B 1993 *An Introduction to the Fractional Integrals and Derivatives—Theory and Applications* (New York: Willey)

[26] Momani S and Odibat Z 2007 Numerical comparison of methods for solving linear differential equations of fractional order *Chaos Solitons Fractals* 1248–55

[27] Odibat Z and Momani S 2006 Application of variational iteration method to nonlinear differential equations of fractional order *Int. J. Nonlinear Sci. Numer. Simul.* 27–34

[28] Oldham K B and Spanier J 1974 *The Fractional Calculus* (New York: Academic)

[29] Rabei E M, Nawafleh K I, Hijjawi R S, Muslih S I and Baleanu D 2007 The Hamilton formalism with fractional derivatives *J. Math. Anal. Appl.* **327** 891–7

[30] Riewe F 1996 Nonconservative Lagrangian and Hamiltonian mechanics *Phys. Rev.* E **53** 1890–99

[31] Riewe F 1997 Mechanics with fractional derivatives *Phys. Rev.* E **55** 3581–92

[32] Sagan H 1992 *Introduction to the Calculus of Variations* (New York: Daver)

[33] Samko S G, Kilbas A A and Marichev O I 1993 *Fractional Integrals and Derivatives—Theory and Applications* (Amsterdam: Gordon and Breach Science Publishers)

[34] Stantilli R M 1978 *Foundations of Theoretical Mechanics I* (Berlin: Springer)

[35] Van Brunt B 2004 *The Calculus of Variations* (New York: Springer)